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LONGEVITY RISK

BY

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Summary

Most of the western world has seen a steady increase in the average lifetime of its inhabitants over the past century. Although the past trends suggest that further changes in mortality rates are to be expected, considerable uncertainty exists regarding the future development of mortality. This type of uncertainty is referred to as longevity risk. This paper reviews the current state of the literature concerning longevity risk. First, we discuss the modeling of future mortality, including the Lee and Carter (J Am Stat Assoc 87:659–671, 1992)-approach, as well as other approaches. Second, we discuss the importance of longevity risk for the solvency of portfolios of pension and life insurance products. Finally, we investigate possibilities for longevity risk management. In particular, we consider longevity risk management through securitization and/or pension and insurance (re)design.

Key words: longevity risk, risk quantification, risk management

1 INTRODUCTION

Most of the western world has seen a steady increase in the average lifetime of its inhabitants over the past century. For example, the expected remaining lifetime of a Dutch male aged 65 increased from 13.5 years in 1975 to 17 years in 2007.[1] The potential effects of trends in mortality on pension costs present significant challenges for governments as well as individual pension funds and life insurers. Biffis and Blake (2009) report that every additional year of life expectancy at age 65 is estimated to add at least 3% to the present value of UK pension liabilities. This clearly illustrates the need to consider interventions that can mitigate the adverse effects on pension and

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1 *Source:* Statistics Netherlands, http://statline.cbs.nl/.

insurance providers, while still guaranteeing an adequate level of retirement and insurance benefits to policyholders. Identifying appropriate interventions is challenging. The major challenge, however, is not in the trend itself, but in the fact that the future development of life expectancy is uncertain. Indeed, although the past trends suggest that further changes in mortality rates are to be expected, there is considerable uncertainty regarding the future development of mortality. Decisions regarding redesign of pension and insurance systems should therefore appropriately account for the effects of this particular uncertainty on the costs of pensions. In addition, since interventions in the design of pension and insurance contracts can mitigate, but not eliminate, the effects of mortality risk, there will be residual risk. Whereas the focus of regulators has long been on the risk in *financial investments*, there is now increasing awareness that accurate quantification and management of the risk in pension and insurance *liabilities* is equally important. For example, the Solvency II project (Group Consultatif Actuariel Europeen 2006), the goal of which is to redesign financial regulation of insurance companies in the EU, has put increased emphasis on the valuation and management of pension and insurance liabilities. Common approaches taken in practice to deal with the effect of changes in life expectancy have included regularly re-estimating the value of the liabilities on the basis of newly estimated death probabilities, or determining the value of the liabilities on the basis of a projected trend in mortality. These approaches, however, are either retrospective, or do not properly account for the uncertainty in the future development of mortality. Risk management practices may need to be adjusted in order to account properly for uncertainty in the future development of mortality.

This paper reviews the literature on longevity risk (i.e., the uncertainty in future changes in mortality rates). The focus is on models to forecast the probability distribution of future mortality rates, approaches to quantify the effect of longevity risk on pension and insurance liabilities, and possibilities for risk management.

The paper is organized as follows. In the next section, we formally define longevity risk, and discuss the distinction to individual mortality risk. We also show that, in contrast to individual mortality risk, longevity risk does not become negligible when portfolio size becomes large. Next, in Section 3 we review the literature on mortality modeling, including the Lee and Carter-approach, which is nowadays used extensively to model the uncertainty in the probability distribution of future mortality. In addition to the original Lee and Carter (1992)-model, we discuss several alternative approaches. Moreover, we decompose longevity risk into process risk and model risk, where the latter includes as special case parameter risk. Model risk arises due to a lack of knowledge regarding the correct probability distribution of future mortality rates, and process risk is the uncertainty in the mortality trends that remains, even in case we exactly would know the correct probability distribution of future mortality rates. Parameter risk is model risk that arises due to sampling inaccuracy, given a selected model (class), like the Lee and Carter-model.

In Section 4, we discuss approaches to quantify the importance of longevity risk for portfolios of (pension) annuities. First, we extend Olivieri (2001) to demonstrate the relative importance of individual mortality risk and longevity risk, and the effect of portfolio size. Second, we discuss results from Hári et al. (2008b) regarding the effect of longevity risk on the volatility of the funding ratio of pension funds. Third, we discuss the approach in Stevens et al. (2010b), who quantify longevity risk by determining its effect on the *probability of ruin*, i.e., the probability that, for a given (re)investment strategy, the current value of the assets will not be sufficient to meet all future liabilities.[2]

Finally, in Section 5 we investigate possibilities for longevity risk management for life insurers and pension funds, following Cairns et al. (2008a). We illustrate some aspects of longevity risk management, in particular, the determination of solvency buffers, and the effect of the product mix as a natural approach to diversify longevity risk. We also briefly discuss the attempts to set up a "life market," a trading place for mortality-based products, that could be used to hedge or to reduce the longevity risk. Section 6 concludes.

2 LONGEVITY RISK

In this section, we first demonstrate the importance of longevity trends for annuity providers. Then, we discuss the distinction between longevity risk and mortality risk, and provide evidence that longevity risk is substantial. Finally, we discuss the implications of longevity risks for pricing annuities (or other longevity related assets and liabilities), as well as for risk management practices.

2.1 *Mortality Trends*

We first introduce some basic terminology and results related to mortality. An important quantity is the "one-year death probability," denoted by $q_{x,t}^{(g)}$, which quantifies at year t the probability that a person of age x and belonging to group g will not survive another year. The probability that the same individual survives at least another year is then given by

$$p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)}. \tag{1}$$

2 We would like to emphasize that these studies not only use different approaches to quantify longevity risk, but also use different models to forecast future mortality. Any difference in the magnitude of longevity risk between these studies can be due to either the choice of method or the choice of forecast model.

If, for example, group g (Dutch males or Dutch females) is understood, we suppress the superindex (g). Moreover, if the probabilities would be independent of time t, we can simplify even further, by writing q_x and p_x. Assuming this for the moment, the probability that the same individual (of age x and belonging to group g, suppressed) survives at least τ more years is then given by

$$_{\tau}p_x = \prod_{j=0}^{\tau-1} p_{x+j}, \tag{2}$$

where $_1p_x = p_x$. Using these probabilities, we can derive e_x, the expected number of years the individual will survive:

$$e_x = \sum_{\tau \geq 1} {}_{\tau}p_x. \tag{3}$$

Thus, seen from year t this individual is expected to die in year $t + e_x$, at age $x + e_x$.

The above, however, assumes that one-year death probabilities are *constant* over time. There is ample evidence that death probabilities change over time. In Figure 1 we plot the one-year death probability $q_{x,t}^{(g)}$ for a number of different ages x and two groups g, namely the group of Dutch males and the group of Dutch females, for the years $t = 1950$ to $t = 2006$, where we normalize by the one-year death probabilities of year $t = 1950$. These one-year death probabilities are obtained from the Human Mortality Database.[3] This figure clearly illustrates that, at least over longer periods, the one-year death probabilities decrease over time, reflecting the increase in longevity over time. But then the assumption that the one-year death probabilities are *constant* over time is not valid. As a consequence, the probability at year t that an individual of age x and belonging to group g survives at least τ other years is no longer given by (2), but, instead, should be calculated as

$$_{\tau}p_{x,t}^{(g)} = p_{x,t}^{(g)} \cdot p_{x+1,t+1}^{(g)} \cdot \cdots \cdot p_{x+\tau-1,t+\tau-1}^{(g)}, \tag{4}$$

using $p_{x+j,t+j}^{(g)} = 1 - q_{x+j,t+j}^{(g)}$; see also (1).

Then, the expected number of years the individual will survive, calculated at year t, is given by

$$e_{x,t}^{(g)} = \sum_{\tau \geq 1} {}_{\tau}p_{x,t}^{(g)}, \tag{5}$$

instead of (3). Thus, to calculate (5), we need *future projections* of the one-year death probabilities $q_{x,t'}^{(g)}$, for $t' \geq t$. Not using such projected one-year

3 See www.mortality.org.

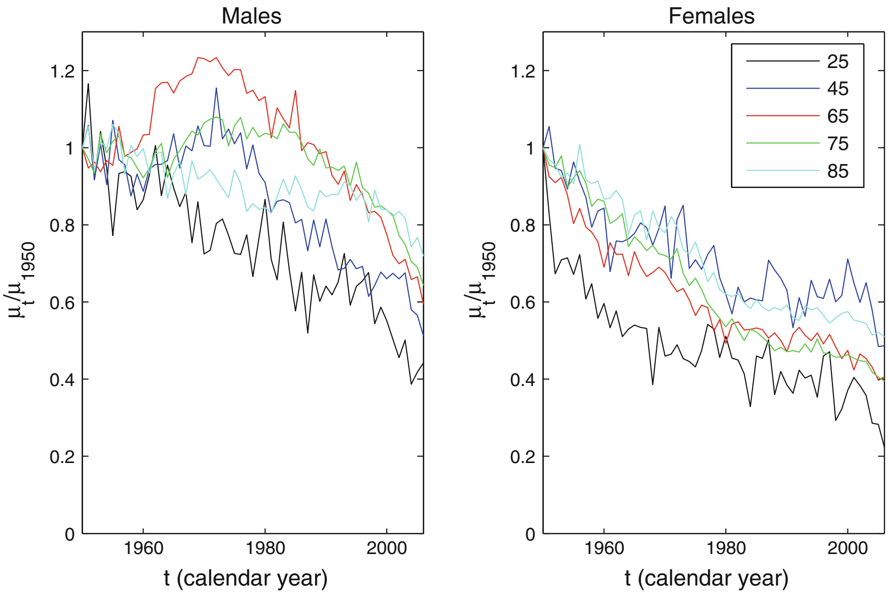

Figure 1 – **One-year death probabilities**. This figure plots the observed one-year death probabilities for Dutch males (*left panel*) and Dutch females (*right panel*), for different ages and for different time periods, normalized to one for the year 1950. The data originates from the Human Mortality Database

death probabilities might result in a serious underestimation of the expected number of years an individual will survive and of the expected discounted value of the annuity. Indeed, Hári et al. (2008b) show that the expected remaining lifetime changes substantially when future changes in mortality rates are taken into account.[4] For the age $x = 65$, they report an increase in the expected remaining lifetime for males from 11.2 years in 1900 to 15.4 years in 2000, and projected to be 16.1 years in 2025, while for females of the same age these numbers are 11.8 years for 1900, 19.4 for 2000, and the projected value for 2025 is 20.6.

Such trends obviously have important implications for the value of pension annuities. As reported in, for instance, Biffis and Blake (2009), every additional year of life expectancy at age 65 is estimated to add at least 3% to the present value of UK pension liabilities. Assuming that such numbers apply more generally, the economic implications of longevity become obvious.

This is confirmed by results from Hári et al. (2008b), who illustrate the effect of longevity trends on the expected present value of annuity payments. Specifically, they consider a (deferred) annuity that pays off one Euro

4 These projections are based on a model, proposed by Hári et al. (2008a).

(in arrears) every year that the annuitant survives, and is older than 65. The expected present value, at time t, for an annuitant aged x belonging to group g is given by:

$$\widetilde{a}_{x,t}^{(g)} = \sum_{\tau \geq \max\{65-x,0\}} {}_{\tau}p_{x,t}^{(g)} \cdot P_t^{(\tau)}, \tag{6}$$

where $P_t^{(\tau)}$ denotes the market value, at time t, of a zero-coupon bond maturing at time $t+\tau$ (i.e., the date-t value of one Euro to be paid in period $t+\tau$). Table 1, taken from Hári et al. (2008b), shows $\widetilde{a}_{x,t}^{(g)}$ as a function of age, for ages varying from 25 to 85 based on so-called period tables (first column), i.e., assuming that $q_{x,t'}^{(g)} = q_{x,t}^{(g)}$, for $t' \geq t = 2004$, and based on forecasted one-year probabilities (second column),[5] for groups of men and women. Comparison of the first and the second columns reveals that the present value of annuity payments based on period life tables underestimates[6] the value based on forecasted death probabilities by 7.7% for a 25-year-old man and 8.8% for a 25-year-old woman. For the 65-year-old, the corresponding numbers are 0.4% and 1.7%, respectively.

2.2 *Sources of Mortality Risk*

While the above illustrates the importance of mortality trends for pension providers, there is at hand a more challenging issue. Indeed, Figure 1 shows not only that the one-year death probabilities (on average) decrease over time, but also that this decrease is different for various ages and different for males and for females in an (at least to some extent) *unpredictable* way. When extrapolating this finding to forecasting future one-year death probabilities, it seems quite implausible to assume that we would be able to know these future one-year death probabilities in a *deterministic* way, without any uncertainty. Instead, it would seem more realistic to deal with this uncertainty, by assuming that the one-year death probabilities $q_{x,t'}^{(g)}$ are *stochastic* at time t, for $t' > t$. If so, we are confronted with *longevity risk*: the probability at year t that an individual of age x and belonging to group g survives at least τ other years (see (4)) is not known deterministically, but is random. The literature therefore distinguishes two sources of mortality risk:[7]

5 In Hári et al. (2008b) longevity risk is already taken into account at this stage, but for pedagogical reasons only we proceed as if the forecasts are deterministic. Hári et al. (2008b) employ a term structure of interest rates calibrated on the interest rates in 2004.

6 There are some exceptions for elderly men, due to the specific forecasted mortality rates employed by Hári et al. (2008b).

7 The literature also distinguishes so-called *mortality catastrophe risk*, which relates to the risk of higher than expected mortality (for example due to an epidemic). The focus in our paper is on individual mortality risk, and, more importantly, longevity risk.

TABLE 1 – **MARKET VALUE OF ANNUITIES**

Age	Men		Women	
	Period table	Projected table	Period table	Projected table
25	0.872	0.944	1.038	1.139
30	1.193	1.279	1.418	1.541
35	1.633	1.733	1.939	2.086
40	2.238	2.350	2.654	2.827
45	3.079	3.198	3.643	3.840
50	4.255	4.373	5.023	5.240
55	5.918	6.022	6.950	7.177
60	8.279	8.356	9.606	9.831
65	10.403	10.441	11.969	12.179
70	8.669	8.677	10.333	10.508
75	6.897	6.881	8.490	8.617
80	5.191	5.151	6.535	6.593
85	3.723	3.675	4.643	4.680

The table shows the market value of the annuity, as a function of age, based on period tables (first column), and based on forecasted mortality rates (second column), for men and for women, where the forecasted mortality rates are assumed to be deterministic (see footnote 5). *Source:* Hári et al. (2008b)

- *Individual mortality risk* refers to the risk due to the fact that, for given death probabilities, an individual's remaining lifetime is a random variable;
- *Longevity risk* refers to the risk as a consequence of *longer term* deviations from deterministic mortality projections.

As a consequence of longevity risk, the expected number of years the individual will survive, calculated at year t (see (5)) becomes random (as well as all other quantities that depend on future one-year death probabilities). Thus, for instance, the above mentioned expected remaining lifetimes taken from Hári et al. (2008b) are just *point estimates*. Figure 2 illustrates the evolution of the probability distribution of the expected remaining lifetimes $e_{x,t}^{(g)}$ for the groups g of Dutch males and Dutch females of age $x=65$, for the years $t=2007$ to 2050, when the future death probabilities are assumed to be random, as will be described in the next section.[8] The graph shows a number of quantiles (ranging from the 0.10- to the 0.90-quantile). The figure shows that there is already substantial longevity risk in the earliest projections

8 We use the quantification described in the appendix of Stevens et al. (2010a,b), which allows for both process and model risk, to be explained in the next section.

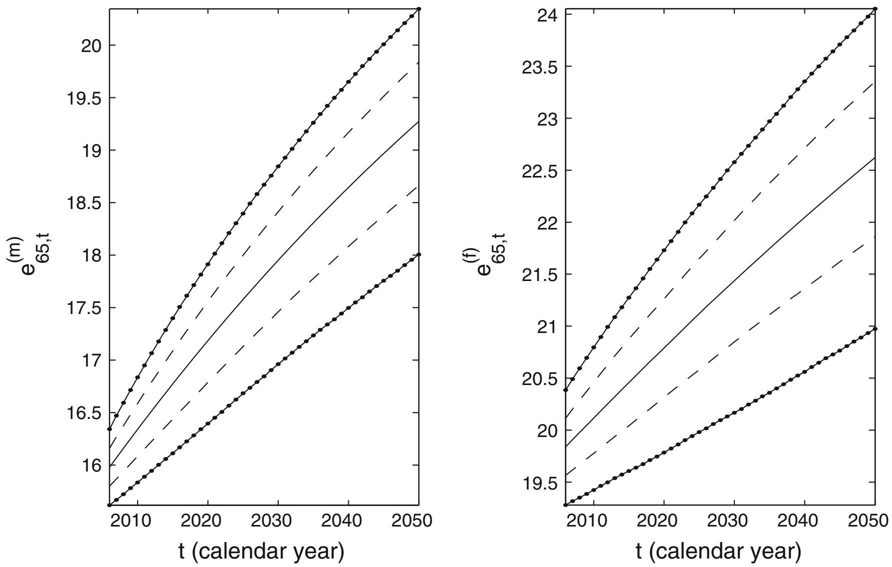

Figure 2 – **Expected remaining lifetimes.** In this figure we plot quantiles (10, 25, 50, 75, 90%) of the distribution of the expected remaining lifetime $e^{(g)}_{x,t}$ for the group g of Dutch males (*left panel*) and Dutch females (*right panel*) of age $x = 65$, for the years $t = 2007$ to 2050. The quantification of the longevity risk is described in Stevens et al. (2010b)

corresponding to $t = 2007$.[9] The quantile intervals for the remaining lifetime of a 65-year old in $t = 2050$ are even much wider.

The significant degree of uncertainty in future expected lifetimes suggests that the effect of uncertain changes in mortality on the value of pension liabilities may also be substantial.

2.3 *On the Importance of Longevity Risk*

In this subsection, we demonstrate that longevity risk, in contrast to individual mortality risk, cannot be diversified away by increasing portfolio size. We discuss the implications for the pricing of longevity-linked assets or liabilities, as well as for the risk management practices of pension funds.

In order to do so, we consider a pool of immediate life annuities sold to N individuals of age x belonging to group g in year t. The annuity pays off one Euro (in arrears) to an individual every year that this individual survives. Assume a constant and risk free annual interest rate r, and denote by $1_{i,t+\tau}$

9 It is a picture similar to that in Dowd et al. (2008), see also Biffis and Blake (2009), who consider the UK population. The figure clearly illustrates that the expected remaining lifetimes of 65-years old are projected to increase in the future.

the dummy variable equal to one in case annuitant i is still alive at time $t+\tau$. Then the present value at time t of the annuity payments to annuitant i in years $t+\tau$, $\tau \geqslant 1$, is given by

$$Y_i = \sum_{\tau \geqslant 1} 1_{i,t+\tau} \frac{1}{(1+r)^\tau}. \tag{7}$$

For the sake of argument, first assume that future death probabilities are known with certainty (i.e., there is individual mortality risk, but no longevity risk). Then, the expected present value at time t of the annuity payments to annuitant i is given by

$$a_{x,t}^{(g)} = \sum_{\tau \geq 1} \mathbb{E}\left[1_{i,t+\tau}\right] \frac{1}{(1+r)^\tau} = \sum_{\tau \geq 1} {}_{\tau}p_{x,t}^{(g)} \frac{1}{(1+r)^\tau}. \tag{8}$$

Using a pooling argument, this expected discounted value is also the fair price of the annuity. The fair price of Y_i will be equal to the fair price of $\frac{1}{N}\sum_{i=1}^N Y_i$. Assume that the Y_i are independent, with expected value $\mu = \mathbb{E}(Y_i)$ and variance $\sigma^2 = \mathrm{Var}(Y_i)$. Then the variance of $\frac{1}{N}\sum_{i=1}^N Y_i$ can be calculated as

$$\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N Y_i\right) = \sigma^2/N. \tag{9}$$

In case N becomes very large, $\frac{1}{N}\sum_{i=1}^N Y_i$ becomes risk free, and its fair price (like the fair price of Y_i) equals its expected discounted value, i.e., there is no risk premium.[10] Thus, the one-year death probabilities $q_{x,t}^{(g)}$, and the corresponding survival probabilities as defined in (1) and (2), represent mortality risk at the individual level, which, however, can be eliminated by an insurance company or pension fund by means of pooling. As a consequence, this individual mortality risk should not be priced.

With longevity risk, however, the fair price of the annuity (and other products with a payoff that depends on future survival outcomes) typically will

10 A no arbitrage argument goes as follows. Let $Y_{i,\tau}$ denote the payoff of the annuity at time $\tau = t' - t$, and let p denote the no arbitrage price of $Y_{i,\tau}$. Suppose M_τ is the relevant Stochastic Discount Factor, such that $p = \mathbb{E}\left(M_\tau Y_{i,\tau}\right)$. Then, assuming that the $M_\tau Y_{i,\tau}$ are identically distributed for different i, we have

$$p = \left(M_\tau Y_{i,\tau}\right) = \mathbb{E}\left(M_\tau\left(\frac{1}{N}\sum_i Y_{i,\tau}\right)\right) = \mathbb{E}(M_\tau)\,\mathbb{E}\left(\frac{1}{N}\sum_i Y_{i,\tau}\right) + \mathrm{Cov}\left(M_\tau, \frac{1}{N}\sum_i Y_{i,\tau}\right).$$

Using $\left|\mathrm{Cov}\left(M_\tau, \frac{1}{N}\sum_i Y_{i,\tau}\right)\right| \leq \sigma(M_\tau)\,\sigma\left(Y_{i,\tau}\right)/N$, we find, for $N \to \infty$, $p = \mathbb{E}\left(Y_{i,\tau}\right)\mathbb{E}(M_\tau) = \mathbb{E}\left(Y_{i,\tau}\right)\frac{1}{(1+r)^\tau}$.

include a (longevity) risk premium. To illustrate this, we return to the annuity portfolio (see (7)). Conditional upon the future death rates at time t, given by the set

$$\mathcal{F}_t = \left\{ q_{x,t+\tau}^{(g)} \mid \tau \geqslant 1 \right\},$$

it still makes sense to assume that the payoffs Y_i are independent, with now mean $\mu(\mathcal{F}_t)$ and variance $\sigma^2(\mathcal{F}_t)$, both depending on $\mathcal{F}_t$. However, when calculating the (unconditional) variance of $\frac{1}{N}\sum_{i=1}^{N} Y_i$ we have to take into account that $\mathcal{F}_t$, the one-year probabilities, are random due to longevity risk. We find

$$\operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \mathbb{E}\left(\operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right)\middle|\ \mathcal{F}_t\right) + \operatorname{Var}\left(\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right)\middle|\ \mathcal{F}_t\right).$$

Therefore,

$$\operatorname{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \mathbb{E}\left(\sigma^2(\mathcal{F}_t)\right)/N + \operatorname{Var}(\mu(\mathcal{F}_t)). \tag{10}$$

The first term on the right hand side (corresponding to the pooling effect) still vanishes with increasing N. However, the second term (reflecting the effect of longevity risk) is independent of N. Thus, with longevity risk, even when N becomes very large, $\frac{1}{N}\sum_{i=1}^{N} Y_i$ does not become risk free anymore. As a consequence, the pooling argument no longer results in an elimination of mortality risk: longevity risk remains, and products whose payoffs depend on future mortality typically will include a (longevity) risk premium. Thus, the expected value $\mathbb{E}(Y_i) = \mathbb{E}(\mu(\mathcal{F}_t))$ may no longer be the fair value of the annuity. We shall refer to this expectation as the *best estimate*. From the point of view of an insurer or pension fund, this best estimate might be seen as a lower bound of the value of the annuity (as a liability).

The result that longevity risk cannot be diversified away using pooling has important implications for both pricing and risk management. First, this non-diversifiability implies that the price of a longevity linked asset or liability is likely to include a (longevity) risk premium. However, annuity payoffs (as well as the payoffs of other products depending on future survival outcomes) typically cannot be hedged by currently traded financial products.[11] As a consequence of this market incompleteness, arbitrage arguments are insufficient to obtain unique market prices of annuities and related products. This seriously complicates the fair valuation of liabilities depending on future survival

11 Sometimes, there are natural hedge possibilities, see, for example, Milevsky and Promislow (2001) or Cox and Lin (2007). See also Section 5.3.

outcomes due to the presence of a (longevity) risk premium. The current literature devotes considerable attention to this pricing problem. Specifically, traditional finance approaches (risk-neutral pricing theories, see, for example, Cairns et al. 2006a) as well as actuarial pricing approaches (Wang's premium principle, see, for example, Lin and Cox 2005) receive considerable attention. However, market incompleteness implies that calibrating these pricing models remains difficult. For a recent and thorough overview of the literature on pricing longevity risk, we refer to Bauer et al. (2010).

Second, non-diversifiability has important implications for risk management. Indeed, the traditional approach used in case of individual mortality risk is to reduce the risk by increasing portfolio size, for example, by mutual reinsurance. As discussed above, however, increasing the portfolio size does not reduce the impact of longevity risk, so that other risk management tools need to be applied. In order to investigate this further, a first important step is the modeling of the probability distribution of future mortality, which we will discuss in the next section. In Section 4, we then illustrate how mortality models can be used to quantify the effect of longevity risk, and evaluate the effectiveness of risk management practices in the presence of longevity risk.

3 MODELING FUTURE MORTALITY

In this section we discuss the quantification of the uncertainty in the probability distribution of future mortality. Reviews of such a quantification include Booth et al. (2006), Pitacco (2004), Tabeau (2001), and the recent monographs by Girosi and King (2008) and Pitacco et al. (2009) . See also Benjamin and Soliman (1993), Delwarde and Denuit (2006), Cairns et al. (2008a) and Hári (2007).

The starting point of the analysis is the *(raw) central death rate*[12] or observed per capita number of deaths, defined by $m_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$, where $D_{x,t}^{(g)}$ denotes the number of people with age x in group g that died in year t, and where $E_{x,t}^{(g)}$ denotes the so-called exposure, being the number of person years in group g with age x in year t. These central death rates are typically observed on a yearly basis, ranging from age $x=0$ to some maximum age, like $x=110$, while the time index t ranges from some starting year, normalized as $t=1$ up to some recent year $t=T$. The number of deaths $D_{x,t}^{(g)}$ and the exposure $E_{x,t}^{(g)}$ can be obtained from population statistics, where the exposure is usually approximated.[13] Given $m_{x,t}^{(g)}$ for all age groups x, one can calculate the one-year death probabilities $q_{x,t}^{(g)}$, see, for example, McCutcheon and Nestbitt

12 Raw refers to as observed in the data.
13 For more details, see, for instance, Gerber (1997) or the technical report corresponding to the Human Mortality Database.

(1973). However, since this is a complicated relationship, one usually makes some additional assumptions to obtain an easier link between $m_{x,t}^{(g)}$ and $q_{x,t}^{(g)}$. For instance, assuming that the exposure is linear in x, results in the relationship

$$q_{x,t}^{(g)} = \frac{m_{x,t}^{(g)}}{1 + \frac{1}{2} m_{x,t}^{(g)}}. \tag{11}$$

Alternatively, one makes assumptions such that the central death rate equals the so-called force of mortality,[14] in which case one obtains

$$q_{x,t}^{(g)} = 1 - \exp\left(-m_{x,t}^{(g)}\right). \tag{12}$$

When quantifying longevity risk, one typically models the evolution of the raw central death rate $m_{x,t}^{(g)}$ or the one year death probabilities $q_{x,t}^{(g)}$ over time for a given group g. In case of the central death rate this results in a decomposition of the raw central death rate in a systematic part, say $\widetilde{m}_{x,t}^{(g)}$, and a remaining idiosyncratic part. The systematic part is then projected into the future, and Equations (11) or (12) are used to find the projected future one-year death probabilities, using the systematic part of the central death rates, instead of the raw versions. In case of the one year death probabilities the modeling will result in a decomposition into a systematic and idiosyncratic part, but now in terms of these one year death probabilities, and again the systematic part ($\widetilde{q}_{x,t}^{(g)}$) is projected into the future. Since the models used to quantify central death rates or the one year death probabilities typically consider a fixed group g, we shall suppress the superindex g in the remainder of this section. In the next subsection, we first briefly review the earlier modeling of mortality. In Section 3.2 we review the Lee and Carter (1992)-approach, while in Section 3.3 we discuss some recent developments.

3.1 *Dynamic Mortality Laws*

For a given time period t, $\widetilde{m}_{x,t}$ or $\widetilde{q}_{x,t}$ might be parameterized in some particular way. Such parameterizations are often called "mortality laws," describing mortality (at time t) as a function of age x. Early mortality laws include the "Gompertz law" (Gompertz 1825), "Makeham's law"(Makeham 1860), and "Thiele's law" (Thiele 1872). A more recent version is the "Heligman-Pollard

14 The force of mortality is defined as $\mu_{x,t}^{(g)} = \lim_{\Delta t \to 0} P\left(0 \le T_{x,t}^{(g)} \le \Delta t\right)/\Delta t$, where $T_{x,t}^{(g)}$ denotes the remaining lifetime at time t of an individual of age x belonging to group g. When the force of mortality is constant within bands of time, i.e., $\mu_{x,t+\tau} = \mu_{x,t}$, for $0 \le \tau < 0$, then the force of mortality equals the central death rate. See, for instance, Gerber (1997) for further details.

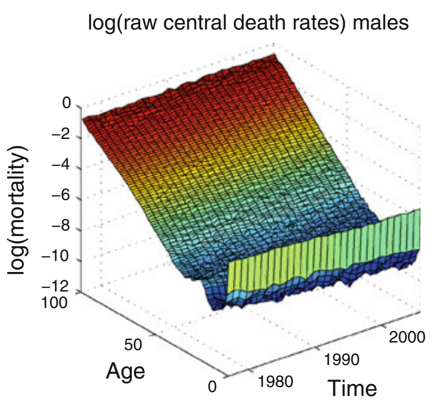

Figure 3 – Log mortality and Lee-Carter fit (Dutch males, using Girosi and King 2006)

law" (Heligman and Pollard 1980), which states (for some given time t, with t suppressed)

$$\widetilde{q}_x = A^{(x+B)^C} + D \exp\left(-E\,(\log x - \log F)^2\right) + \frac{GH^x}{1+GH^x}, \tag{13}$$

where A-H are (unknown) parameters. This law consists of three components, the first of which aims to capture infant and childhood mortality, the second one adult mortality,[15] and the third one the mortality of the elderly.

An obvious way to obtain dynamic mortality models, is to fit some given mortality law each period t for which data is available, with some or all parameters time dependent. The resulting time series of time-dependent parameter values can then be quantified using appropriate statistical or econometric models. Using such models makes forecasting future mortality trends as well as quantifying longevity risk a straightforward exercise, at least, theoretically. However, as argued by, for instance, Tabeau (2001), fitting mortality laws per period with time dependent parameters, typically generates rather unstable results, making forecasting mortality trends using this approach from a practical point of view quite difficult, if not impossible. One way to avoid the instability is to combine a mortality law with age and time dependent polynomials, see, for instance, Renshaw et al. (1996). Using polynomials of sufficient order allows quite an accurate in-sample fit. However, using higher order polynomials to make out-of-sample forecasts typically does not work well, see, for example, Bell (1984) for further clarification.

15 More precisely, the so-called "accident hump," see Figure 3.

3.2 *The Lee and Carter Approach*

Lee and Carter (1992) propose a parsimonious dynamic mortality model that turned out to perform quite well. The model postulates

$$\ln\left(m_{x,t}\right) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{14}$$

with time-independent parameters α_x and β_x, and a (white noise) error term $\epsilon_{x,t}$, where $\{\kappa_t\}$ is a one-dimensional underlying time-dependent latent process that quantifies the evolution of mortality over time. The parameter α_x quantifies the level of the log central death rate of age x, while the parameter β_x quantifies the age x-specific sensitivity of the log central death rate to changes in the group-wide evolution (improvement) as represented by κ_t. The error term $\epsilon_{x,t}$ captures the age and time specific variations around the systematic trend. Due to lack of identification, Lee and Carter (1992) normalize by setting $\sum_x \beta_x = 1$ and $\sum_t \kappa_t = 0$, where the first sum is over all available ages and the second sum over all time periods available in the sample.

Lee and Carter (1992) proposed estimating the model in three steps. In the first step, Singular Value Decomposition (SVD) is applied to find the unique least squares solution (given the normalizations) yielding $\{\widehat{\kappa}_t\}$, $\{\widehat{\alpha}_x\}$, and $\left\{\widehat{\beta}_x\right\}$. The estimated $\{\widehat{\kappa}_t\}$ are then adjusted to ensure equality between the observed and model-implied number of deaths in a certain period (i.e., $\{\widehat{\kappa}_t\}$ is replaced by $\{\widetilde{\kappa}_t\}$) such that

$$\sum_x D_{x,t} = \sum_x \left[E_{x,t} \exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_t\right)\right], \tag{15}$$

with $D_{x,t}$ the number of deaths and $E_{x,t}$ the exposure, introduced at the beginning of this section. This readjustment is done in order to avoid sizeable differences between the number of observed deaths and the model-implied number of deaths. The systematic part, defined as $\widetilde{m}_{x,t} = \exp\left(\alpha_x + \beta_x \kappa_t\right)$, is estimated by $\widehat{\widetilde{m}}_{x,t} = \exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_t\right)$. Finally, the Box-Jenkins method is used to identify and estimate the dynamics of the latent factor $\widetilde{\kappa}_t$. Lee and Carter (1992) find as a process for the dynamics of the latent factor a random walk with drift, i.e.,

$$\kappa_t = c + \kappa_{t-1} + \delta_t, \tag{16}$$

with c the drift term, and with $\{\delta_t\}$ a white noise process, assumed to follow a normal distribution with mean zero and variance equal to σ_δ^2. The parameters c and σ_δ^2 can be estimated applying standard statistical or econometric time-series techniques.

To avoid the second step of this three-step procedure, Wilmoth (1993) proposed a weighted Singular Value Decomposition. In addition, Lee and Miller (2001) proposed replacing the matching according to Equation (15) by a matching on the basis of observed and modeled life expectancy. Moreover, these authors suggest restricting the sample period to a recent time period, in order to avoid a potential misspecification due to a violation of the assumption of constant α_x and β_x. Booth et al. (2002) suggest using statistical techniques to select an appropriate sample period, in line with the assumption of constant α_x and β_x.

The Lee and Carter (1992)-model can easily be extended to include more time factors (in addition to κ_t) (see Renshaw and Haberman (2003a)). However, Tuljapurkar et al. (2000), investigating the G7 countries (Canada, France, Germany, Italy, Japan, UK, and US),[16] find that a single factor (as in the original Lee and Carter (1992) specification) already suffices to explain over 94% of the variance in the log-specific raw central death rates. Nevertheless, to improve the forecast performance, it might be better to include an additional cohort-specific factor (see Renshaw and Haberman 2006).

Mortality projections can be obtained by first predicting future values $\widetilde{\kappa}_{T+t}$ (with T the final year of the sample), then predicting the systematic part of the future central death rates as

$$\widehat{\widetilde{m}}_{x,T+t} = \exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_{T+t}\right), \tag{17}$$

and, finally, calculating the corresponding projected future one-year death probabilities $q_{x,T+t}$, using Eq. (11) or (12). Alternatively, Lee and Miller (2001) suggest predicting the future central death rates $\widetilde{m}_{x,T+t}$ using the observed (raw) central death $m_{x,T}$ of the final year in the sample as a jump-off value, i.e., to calculate

$$\widehat{\widetilde{m}}_{x,T+t} = m_{x,T} \exp\left(\widehat{\beta}_x \left(\widetilde{\kappa}_{T+t} - \widetilde{\kappa}_T\right)\right). \tag{18}$$

In this way, a jump-off bias can be avoided.

Longevity risk arises, first of all, due to the random character of $\widetilde{\kappa}_{T+t}$, whose exact values are of course unknown at time T, even if its distribution function would be exactly known. This longevity risk is referred to as *process risk*. In addition, there is *model risk*: since we do not know the exact distribution of $\widetilde{\kappa}_{T+t}$, we have to model it, possibly incorrectly, which generates model risk. In particular, if we estimate the probability distribution of $\widetilde{\kappa}_{T+t}$, like in the Lee and Carter-approach, there is model risk due to the sampling error in the estimated parameters $\widehat{\alpha}_x$, $\widehat{\beta}_x$, for all ages x, and in the estimates of the drift term c and variance σ_δ^2 of the random walk. This particular model risk is referred to as *parameter risk*.[17] To quantify these risks, Lee and

16 For a more recent multi-country comparison of various stochastic mortality models, see, for example, Booth et al. (2006).

17 There are other sources of model risk as well. For instance, the Lee and Carter (1992) model class might be too small, not containing the actual distribution of $\widetilde{\kappa}_{T+t}$. This is also

Carter (1992) suggest using a bootstrap method. They focus on the parameter risk in the time process (16) only, arguing that the parameter risk in α_x and β_x is small. Koissi et al. (2006) extend the bootstrap procedure to include all parameter risk. See also Renshaw and Haberman (2008).

The longevity risk, which consists of process and parameter risk, can be illustrated by a reformulation of the Lee and Carter (1992)-model by Girosi and King (2006). First, let

$$\ell_t = \begin{pmatrix} \ln\left(m_{1,t}\right) \\ \vdots \\ \ln\left(m_{ma,t}\right) \end{pmatrix}, \tag{19}$$

with ma the maximum age considered; similarly, let $\alpha = (\alpha_1, \ldots, \alpha_{ma})'$, $\beta = (\beta_1, \ldots, \beta_{ma})'$, and $\epsilon_t = \left(\epsilon_{1,t}, \ldots, \epsilon_{ma,t}\right)'$. Then, using (16),

$$\begin{aligned} \ell_t &= \alpha + \beta\kappa_t + \epsilon_t \\ &= \beta c + \left(\alpha + \beta\kappa_{t-1} + \epsilon_{t-1}\right) + \left(\beta\delta_t + \epsilon_t - \epsilon_{t-1}\right) \\ &= \theta + \ell_{t-1} + \zeta_t \end{aligned} \tag{20}$$

with

$$\theta = \beta c, \;\; \zeta_t = \beta\delta_t + \epsilon_t - \epsilon_{t-1}.$$

The Lee and Carter (1992)-model rewritten in this way can easily be estimated and used to make predictions and to quantify the longevity risk. For instance, with $\Delta\ell_t = \ell_t - \ell_{t-1}$, we can estimate θ simply by the time average of $\Delta\ell_t$, i.e., by

$$\widehat{\theta} = \frac{1}{T-1}\sum_{t=2}^{T}\Delta\ell_t = \frac{1}{T-1}\left(\ell_T - \ell_1\right). \tag{21}$$

This estimator has well-known (T-asymptotic) characteristics (depending on the distributional assumptions imposed on ζ_t), implying that making predictions as well as quantifying the longevity risk becomes a standard exercise in statistics or econometrics (both theoretically and practically). In Figure 3 the left panel shows the logarithm of the raw central death rates of Dutch

Footnote 17 continued
a source of model risk: we might require a more extensive or an other model class than the Lee and Carter (1992)-model class if we want to include the actual distribution of $\widetilde{\kappa}_{T+t}$. Possible other model classes, corresponding to much more (model) longevity risk, are discussed in the next subsection. However, the limited availability of mortality data makes it quite hard to determine whether the Lee and Carter (1992) model class is large enough or not. In this paper, we focus on model risk within the Lee and Carter (1992)-model class.

males for ages 0 to 99 years and age class 100–110 years (indicated as age 100), over the sample period 1977 to 2006. This graph shows for each year the typical pattern of mortality as a function of age, starting rather high at age zero, revealing the level of infant mortality, then going down rather steeply to around age 10, and then increasing slowly, with a hump around age 20. This hump is typical for Dutch males, absent in the similar graph for Dutch females.[18] The right panel of Figure 3 shows the fitted values of the Girosi and King (2006)-variant of the Lee and Carter (1992) model, showing that this parsimonious model seems to be able to fit the mortality patterns observed in the data quite well.

Next, we illustrate in Figure 4 the 30 year ahead prediction of the logarithm of the central death rates for 65 year old Dutch males and females, using the Girosi and King (2006) -variant of the Lee and Carter (1992) model. The prediction starts at the year 2007, the first year after the available sample period. In this figure we also include longevity risk, distinguishing between only process risk and the combination of process and parameter risk (in both cases 95% confidence intervals). The graphs show a clear estimated downward trend, both in-sample and predicted out-of-sample. In case of 65 year old males this trend corresponds to a decrease of the one-year death probability[19] of 0.0269 at the beginning of the sample (1977) down to 0.0141, predicted 30 years ahead, a decrease of almost 50%. In case of females, the one year death probability in 1977 equals 0.0120 and is predicted to go down to 0.0084, predicted 30 years ahead, a decrease of around 30%. However, these predictions are surrounded with substantial longevity risk (consisting of both process and parameter risk), including (with 95% confidence according to the model) no further decrease in mortality as well as a much more steeper decrease than during the sample period.

3.3 *Recent Dynamic Mortality Models*

The number of deaths is an integer-valued variable. Therefore, a Poisson process might be a more plausible way to model the number of deaths. Brouhns et al. (2002a) model the integer-valued number of deaths $D_{x,t}$ as a Poisson distributed random variable,

$$D_{x,t} \sim \text{Poisson}\left(E_{x,t}\widetilde{m}_{x,t}\right), \tag{22}$$

with the systematic part of the central death rate $\widetilde{m}_{x,t}$ modeled as $\widetilde{m}_{x,t} = \exp\left(\alpha_x + \beta_x \kappa_t\right)$, comparable to the Lee and Carter (1992)-model. The model can be estimated following the same steps as in the original Lee and Carter

18 In case of other countries, this hump is typically observed for *both* males and females.

19 Calculated using Eq. (12).

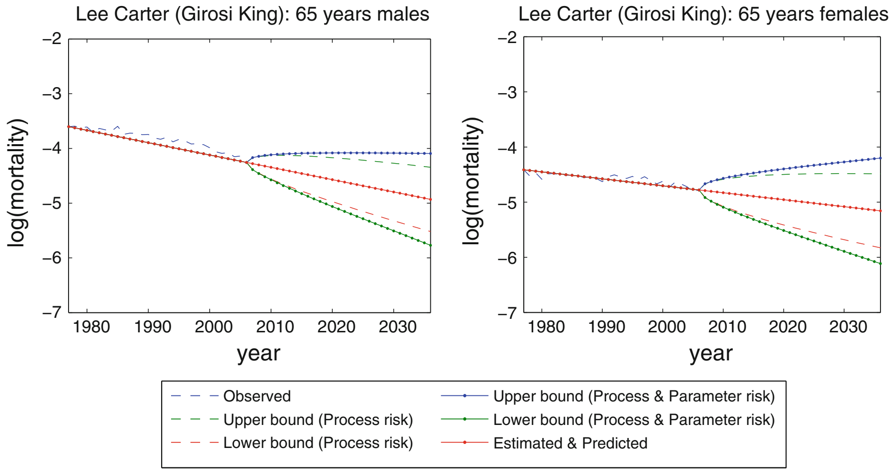

Figure 4 – Predicting log mortality (using Girosi and King (2006) -variant of Lee-Carter)

(1992)-approach, but with the first step replaced by maximum likelihood using, for instance, the iterative method proposed in Goodman (1979). Brouhns et al. (2005) discuss bootstrapping the Brouhns et al. (2002a)-model in order to quantify the longevity risk.

Cossette et al. (2007) propose as adjustment of the Lee and Carter (1992)-model to model the number of deaths as a Binomial process

$$D_{x,t} \sim \text{Bin}\left(E_{x,t}, q_{x,t}\right), \tag{23}$$

with $q_{x,t}$ modeled as $q_{x,t} = 1 - \exp\left(-\widetilde{m}_{x,t}\right)$, according to Eq. (12). The systematic part of the central death rate (or force of mortality) is again modeled in line with Lee and Carter (1992) as $\widetilde{m}_{x,t} = \exp\left(\alpha_x + \beta_x \kappa_t\right)$. This model can be estimated like the Brouhns et al. (2002a)-model, and the longevity risk can be quantified by means of bootstrapping.

The Lee and Carter (1992)-model implicitly assumes that there is no heterogeneity in the measurement error terms $\epsilon_{x,t}$, see (14). Li et al. (2006) propose a way to incorporate heterogeneity into the Brouhns et al. (2002a)-variant of the Lee and Carter (1992)-model. Alternatively, Delwarde et al. (2007) suggest to use the Negative Binomial distribution to allow for more heterogeneity.

The Lee and Carter (1992)-model results in estimates for the parameters α_x and β_x for each given age x. Using α_x and β_x for each year of age might result in localized age induced anomalies. Lee and Carter (1992) proposed to have age groups $\left([0,\ 1),\ [1,\ 5),\ [5,\ 9)\ \ldots,\ [80,\ 85)\right)$, and in addition the age group [85, 109). Such age groups avoid localized age induced anomalies. However, this method leads to mortality rates that are equal for age groups of five years. Such an approximation might be quite crude, especially for valuating pension contracts. Renshaw and Haberman (2003b) propose to first estimate the parameters of the model using the one-year age groups and then to smooth using, for instance, a cubic spline. More recently, Delwarde et al. (2007) propose to smooth the β_x parameters as part of the first step, using a penalized log-likelihood approach in the Brouhns et al. (2002a)-variant of the Lee and Carter (1992)-model.

The Lee and Carter (1992)-approach also has some drawbacks. An important drawback follows from the reformulation by Girosi and King (2006). As follows from the estimator (21), see also Figure 4, the drift term of the random walk can be estimated by fitting a line for each age x through the first and final observation of the $\ln\left(m_{x,t}\right)$ in the sample. Extrapolating these lines yields the age specific mid-points of the mortality projections (the "point estimates"). However, as long as the lines corresponding to different ages are not parallel, this implies that (very) long term mortality projections might become quite implausible, as is clearly illustrated in Girosi and King (2006), see also Girosi and King (2008). Their solution is to work with appropriate priors.

The problem of deviating long term forecasts might become even worse when the Lee and Carter (1992) methodology is applied to different groups g, each with its own specific process $\{\kappa_t^{(g)}\}$, representing the evolution of mortality over time. However, Wilson (2001) documents a global convergence in mortality levels. Li and Lee (2005) propose to adapt the Lee and Carter (1992)-approach by first identifying the central tendency, resulting in a common random walk with drift process $\{\kappa_t\}$, representing the joint evolution over time, and then to find the group specific stationary time processes $\{\kappa_t^{(g)}\}$, that represent the short term group g deviations from the common time trend.

Finally, the Lee and Carter (1992)-model can only be used for groups for which sufficient data on mortality of different ages is available. Typically, this is an entire population of males and/or females of a country or a large region. However, the relevant population for an insurance company or a pension fund might deviate from the population for which data is available. For instance, Brouhns and Denuit (2002) and Denuit (2008) find that there is a significantly lower mortality rate for the group of insured individuals that were investigated compared with the whole male and female Belgian population. This might limit the applicability of the Lee and Carter (1992)-approach. Plat (2008) proposes a way to construct a portfolio-specific stochastic mortality model.

Next to the Lee and Carter (1992)-time series based stochastic mortality models, there are also other classes of time series based stochastic mortality models, for instance, imposing extra smoothness. Cairns et al. (2006b) propose a model that builds in smoothness in mortality rates across adjacent ages in the same year. Currie et al. (2004) propose a model assuming smoothness across both ages and years. Cairns et al. (2007, 2008b,c) provide an extensive comparison of these various time series based stochastic mortality models.

To illustrate the effect of smoothing we present in Figure 5 an application of the Currie et al. (2004)-method in terms of the logarithm of the central death rate, using the same data as in case of Figs. 3–4. The upper panels contain the in-sample results for 65 year old males (left) and females (right). These graphs show that the evolution of mortality over time has some curvature, which is captured quite well and in a smooth way by the Currie et al. (2004)-method (which employs so-called B-splines). Such a curvature will not be captured by the Lee and Carter (1992) model. In case of males there seems to be some acceleration in the decrease of mortality, while for females there is at first some slowing down and then again a slight acceleration in the decrease of mortality. The lower panels show the 30 years ahead predictions including 95% confidence intervals reflecting the longevity risk. The acceleration with regard to the males is translated into forecasts that are much lower than those derived from the Girosi and King (2006)-variant of the Lee and Carter (1992) model. In fact, 65 year old males and females are predicted to have more or less the same mortality characteristics 30 years from now. However, the longevity risk is quite substantial, leaving the possibility (with 95% confidence according to the model) of a wide variety of possible future mortality trends. The result of much wider prediction intervals, when changing the model from Lee and Carter (1992) to Currie et al. (2004), shows the importance of taking into account model risk.

4 QUANTIFYING LONGEVITY RISK

There are several studies that illustrate the importance of longevity risk for pension funds and insurance companies. The approaches differ both in terms of how longevity risk is quantified, and in terms of how the probability distribution of future mortality is modeled. For the former, we distinguish three approaches. First, an often used approach to quantify longevity risk in annuity portfolios is to determine its effect on the probability distribution of the present value of all future payments, for a given, deterministic, and constant term structure of interest rates (see, for example, Olivieri 2001, Brouhns et al. 2002b , Dowd et al. 2006, and Cossette et al. 2007). Next, there is some literature that focuses on the effect of longevity risk on a pension fund's probability of underfunding (Olivieri and Pitacco 2003, Hári et al. 2008b). Finally, longevity risk can be quantified by determining its effect on the probability of ruin for a portfolio of longevity-linked liabilities (Olivieri and Pitacco 2003, Stevens et al. 2010b).

With regard to modeling the probability distribution of future mortality, several approaches discussed in the previous section are used, with the most

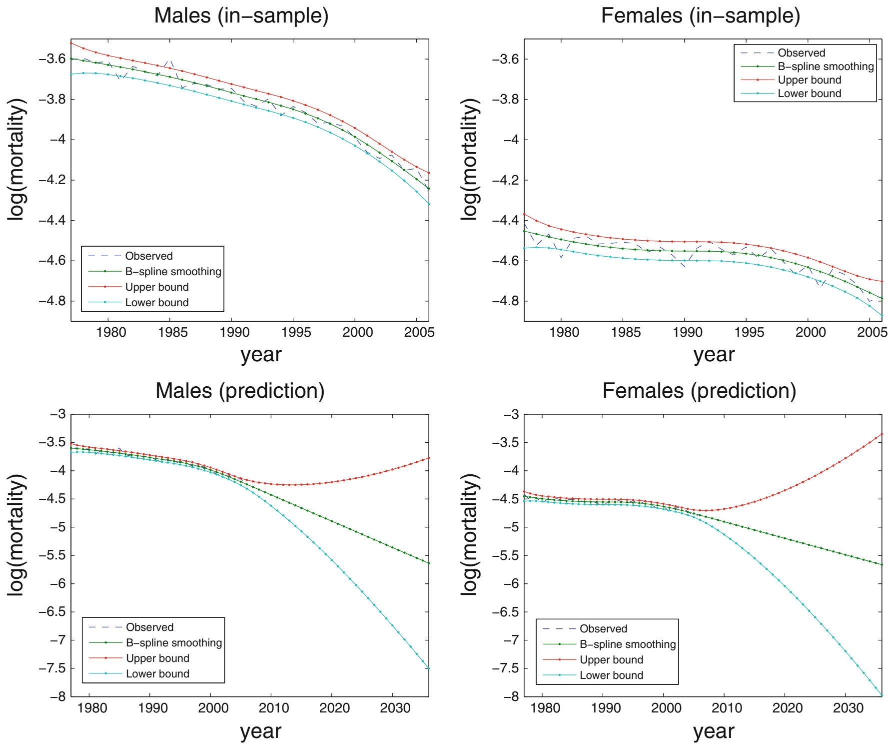

Figure 5 – Smooth log mortality estimation and prediction (using Currie et al. (2004))

popular among them the variants of the Lee and Carter (1992)-approach. For example, Brouhns et al. (2002a) use the variant with the Poisson distribution, and Cossette et al. (2007) use the variant with the Binomial distribution. Olivieri (2001) and Milevsky et al. (2006) instead present theoretical studies showing the implications of longevity risk in a setting where uncertainty in future mortality is modeled by means of three hypothetical scenarios. Other illustrations and references can be found in the review articles and monographs, mentioned at the beginning of the previous section.

In Sections 4.1, 4.2, and 4.3, we discuss the approaches in Olivieri (2001), Hári et al. (2008b), and Olivieri and Pitacco (2003), respectively. In each case, we consider a given and fixed date t, and quantify the effect of longevity risk on the liability payments in all future years.

Throughout this and the following section, we denote BEL_τ for the *best estimate value* of the liabilities at date $\tau \geq t$, which is defined as the market value of the liabilities in the best estimate scenario for future mortality development, i.e.,

$$BEL_\tau := \sum_{s \geq 1} \mathbb{E}_\tau \left[\widetilde{L}_{\tau+s} \right] \cdot P_\tau^{(s)}, \tag{24}$$

where $\widetilde{L}_{\tau+s}$ denotes the liability payment at time $\tau + s$, $P_\tau^{(s)}$ denotes the date-τ market value of a zero-coupon bond maturing at time $\tau + s$, and $\mathbb{E}_\tau[\cdot]$ denotes the expectation, conditional on death rates up to time τ.

4.1 *Discounted Present Value of Liabilities*

In this subsection we discuss the analysis in Olivieri (2001), who focusses on the relative importance of individual mortality risk and longevity risk. She quantifies longevity risk in annuity portfolios by determining its effect on the probability distribution of the present value of future payments. The one-year death probabilities are assumed to follow the mortality law of Heligman-Pollard. Olivieri (2001) incorporates longevity risk by considering three possible future scenarios (a worst case, a medium case, and a best case). In this subsection, we replicate her results, but instead of assuming three possible scenarios in terms of the Heligman-Pollard mortality law, we model the uncertainty in the probability distribution of future mortality following Stevens et al. (2010a). This means that we include process and parameter risk in the future death probabilities on the basis of the Lee and Carter (1992)-approach. In addition, we shall allow for uncertainty in the model-variant choice: We include next to the traditional Lee and Carter (1992)-model also the variants proposed by Brouhns et al. (2002a) and Cairns et al. (2007). In this way we also allow for model risk.[20]

For a given and fixed year t, we consider the present value of all future payments in a portfolio of pension annuities. There are N annuitants, all of age $x = 65$ at time t. In our case t corresponds to the year 2006. The annuity pays off one Euro every year that the annuitant survives. The time-t present value of the annuity payments to annuitant i, denoted by Y_i, is defined in (7), where we shall assume a constant annual interest rate, equal to $r = 0.04$. Conditional upon the one year death probabilities after time t, we can calculate the expected value of the present value Y_i, for a given i, resulting in $a_{65,t}^{(m)}$ for the Dutch male and in $a_{65,t}^{(f)}$ for the Dutch female, where $a_{x,t}^{(g)}$ is defined in Eq. (8). Without longevity risk, this expectation would be the fair value of the annuity. In Figure 6 we present the distributions of $a_{65,t}^{(m)}$ and $a_{65,t}^{(f)}$, when the future death probabilities are random as described above (including process, parameter, and model risk). The distribution for females (around just below 13 Euro) is shifted to the right compared to the distribution of males

20 For a detailed description we refer to the appendix of Stevens et al. (2010a or 2010b).

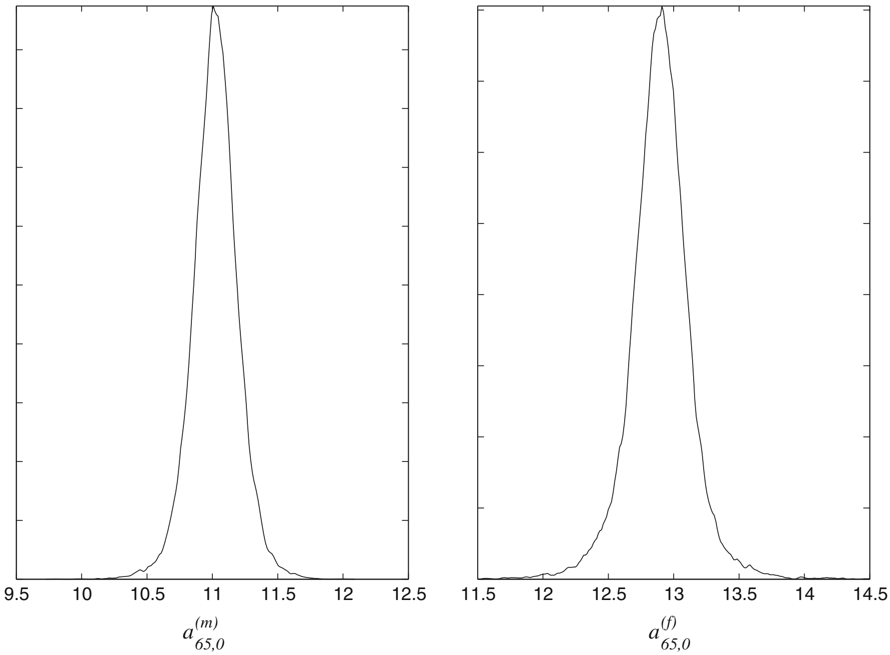

Figure 6 – Distribution annuity portfolio. This figure presents the distribution of the annuity portfolio at time $t=0$ (corresponding to the year 2006) due to longevity risk only (i.e., after pooling). For all annuitants the age is $x=65$. The *left panel* applies to a portfolio of males, the *right panel* to a portfolio of females

(around 11 Euro). This reflects the fact that females, on average, become older than males. Moreover, the figure clearly reveals substantial longevity risk in the annuities, implying that a fair valuation might require a substantial risk premium.

Next, we reproduce Table 5 of Olivieri (2001). The present value of the portfolio of annuities is given by $\mathcal{Y}=\sum_{i=1}^{N} Y_i$. We shall assume that conditional upon $\mathcal{F}_t=\left\{q^{(g)}_{x,t+\tau} \mid \tau \geqslant 0\right\}$ the Y_i are distributed independently. Table 2 presents our results.

The first row reports the best estimates of the annuity portfolio, for both males and females for different sizes N. For $N=1$ it yields the best estimate for the annuity. The next three rows present the variances of these portfolios, together with a decomposition as in Eq. (10):

$$\mathrm{Var}(\mathcal{Y})=\mathbb{E}(\mathrm{Var}(\mathcal{Y} \mid \mathcal{F}_t))+\mathrm{Var}(\mathbb{E}(\mathcal{Y} \mid \mathcal{F}_t)). \tag{25}$$

The first term on the right hand side of this equation corresponds to the portfolio risk if there would be no longevity risk. This risk increases linearly

TABLE 2 – DESCRIPTIVE STATISTICS ANNUITY PORTFOLIO

	Males			Females		
	$N=1$	$N=100$	$N=1000$	$N=1$	$N=100$	$N=1000$
$\mathbb{E}(\mathcal{Y})$	11.024	1102.365	11023.654	12.897	1289.737	12897.374
$\mathbb{E}(\text{Var}(\mathcal{Y}\|\mathcal{F}_t))$	19.604	1960.397	19603.968	17.963	1796.332	17963.316
$\text{Var}(\mathbb{E}(\mathcal{Y}\|\mathcal{F}_t))$	0.031	312.609	31260.878	0.057	567.703	56770.266
$\text{Var}(\mathcal{Y})$	19.635	2273.006	50864.845	18.020	2364.034	74733.582
γ	0.40197	0.04325	0.02046	0.32914	0.0377	0.0212

in N, since $\mathbb{E}(\text{Var}(\mathcal{Y} \mid \mathcal{F}_t)) = N\mathbb{E}(\text{Var}(Y_i \mid \mathcal{F}_t))$. The second term on the right hand side of (25) is due to the presence of longevity risk. This term increases by N^2 with increasing portfolio size N, since $\text{Var}(\mathbb{E}(\mathcal{Y} \mid \mathcal{F}_t)) = N^2\text{Var}(\mathbb{E}(Y_i \mid \mathcal{F}_t))$. Thus, for larger portfolios this term will dominate the total portfolio risk. This can also be seen from the results presented in the table.

The final row of the table presents the coefficient of variation of $\mathcal{Y}$, defined by $\gamma = \sqrt{\text{Var}(\mathcal{Y})}/\mathbb{E}(\mathcal{Y})$. This coefficient allows a better investigation of the size of the portfolio on its riskiness. Without longevity risk, this coefficient vanishes with increasing portfolio size N, due to the pooling effect. However, with longevity risk, we get

$$\gamma = \left(\frac{1}{N} \frac{\mathbb{E}(\text{Var}(Y_i \mid \mathcal{F}_t))}{\mathbb{E}(Y_i)} + \frac{\text{Var}(\mathbb{E}(Y_i \mid \mathcal{F}_t))}{\mathbb{E}^2(Y_i)} \right)^{1/2}, \tag{26}$$

showing that for large portfolio sizes N indeed the longevity risk dominates the total risk, and also does not disappear. In our example, the limiting value of the coefficient of variation equals $\gamma = 0.0160$ for males and $\gamma = 0.0185$ for females.

Olivieri (2001) also calculates the boundary portfolio size $\overline{N}$ such that for portfolio sizes larger than this bound longevity risk dominates the total risk. She calculates this bound as

$$\overline{N} = \frac{\mathbb{E}(\text{Var}(\mathcal{Y} \mid \mathcal{F}_t))}{\text{Var}(\mathbb{E}(\mathcal{Y} \mid \mathcal{F}_t))}. \tag{27}$$

In our case (i.e., with survival probabilities forecasted with the Lee and Carter-methodology), the boundary value is $\overline{N} = 628$ for males and $\overline{N} = 317$ for females. Although substantially larger than the $\overline{N} = 12$ reported by Olivieri (2001), these numbers are quite low, indicating that also for smaller portfolio sizes longevity risk is an important risk that should be taken into account in a risk management framework.

4.2 *Funding Ratio Volatility*

A drawback of the approach described in Section 4.1 is that it is a "liability only" approach: it ignores the potential impact of financial risk on the importance of longevity risk. Therefore, in this subsection we discuss an alternative approach in which the importance of longevity risk is quantified by determining its effect on the probability distribution of the funding ratio at a future date (see, for example, Olivieri and Pitacco 2003, and Hári et al. 2008b). The funding ratio is defined as the value of the assets divided by the value of the liabilities. Determining the value of longevity-linked liabilities, however, is still a contentious issue. There is extensive literature on the pricing of longevity-linked liabilities (see, for example, Bauer et al. 2010), but due to the high degree of illiquidity and market incompleteness, it remains difficult to calibrate these pricing models. Therefore, the regulator requires that the liabilities should be valued at so-called *fair value*.

Hári et al. (2008b) use a simulation analysis to determine the distributional characteristics of the funding ratio at the beginning of year $t+T$, for maturities $T=1$ and $T=5$, respectively, given that the funding ratio in year t equals 1. They consider a pension fund with N annuitants at the beginning of year $t=2004$, and quantify the uncertainty in future funding ratios for various investment strategies. In order to illustrate the effect of portfolio size, they consider portfolios of different sizes. In each case, the age and gender composition of the pension fund is the portrayal of the Dutch population at the beginning of 2004. An annuitant aged x has built up the right to receive a normalized annual old-age payment of $\min\left\{\frac{x-25}{40}, 1\right\}$ as of the age of 65. They use a *run-off approach* (i.e., they consider a setting where there are no new entrants into the fund, and no rights are built up or premiums are paid after time t), and let the fair value of the liabilities be given by the best estimate value, as defined in (24).[21]

Table 3 shows the simulated distributional characteristics of the funding ratio at time $t+T=t+5$, for five investment strategies: (a) liabilities are 'perfectly' hedged: expected liabilities are hedged with cash-flow matching initially; (b) liabilities are duration hedged, based on the McCauley duration; (c) assets are invested exclusively in 5-year bonds; (d) 50% of the assets is invested into 5-year and 50% in 10-year bonds; (e) 37.5% is invested into 5-year, 37.5% in 10-year bonds, and the rest is invested into stocks; (f) 25% is invested in 5-year, 25% in 10-year bonds, while the rest is invested in stocks. Because individual mortality risk becomes negligible when the portfolio size is infinitely large, the fourth column yields the effect of different investment strategies on funding ratio uncertainty in absence of longevity risk.

21 This is in line with Dutch solvency regulations at the time the research was performed.

TABLE 3 – DISTRIBUTION OF FUTURE FUNDING RATIO WITH MARKET RISK AND LONGEVITY RISK COMBINED, $T=5$

		NL population						
		Micro				Micro + Macro + Parameter		
		500	5000	10000	Infinity	500	5000	10000
Perfect hedge of market risk	$StDev[FR_T]/E[FR_T]$	0.023	0.007	0.005	0.000	0.058	0.053	0.053
	Q(0.025)	0.959	0.986	0.991	1.000	0.901	0.910	0.911
	Q(0.975)	1.048	1.015	1.010	1.000	1.120	1.113	1.113
	$E[FR_T \mid FR_T<Q(0.025)]$	0.953	0.984	0.989	1.000	0.888	0.898	0.899
	$E[FR_T \mid FR_T>Q(0.975)]$	1.058	1.017	1.012	1.000	1.144	1.130	1.130
Static Duration hedge	$StDev[FR_T]/E[FR_T]$	0.038	0.032	0.031	0.031	0.069	0.065	0.064
	Q(0.025)	0.919	0.930	0.931	0.931	0.872	0.878	0.877
	Q(0.975)	1.065	1.053	1.051	1.051	1.137	1.124	1.122
	$E[FR_T \mid FR_T<Q(0.025)]$	0.903	0.916	0.916	0.917	0.854	0.863	0.864
	$E[FR_T \mid FR_T>Q(0.975)]$	1.081	1.064	1.062	1.061	1.159	1.147	1.148
100%-0%-0%	$StDev[FR_T]/E[FR_T]$	0.033	0.024	0.024	0.023	0.062	0.057	0.057
	Q(0.025)	0.954	0.967	0.968	0.968	0.906	0.916	0.915
	Q(0.975)	1.082	1.063	1.062	1.061	1.148	1.137	1.137
	$E[FR_T \mid FR_T<Q(0.025)]$	0.943	0.957	0.957	0.959	0.891	0.899	0.900
	$E[FR_T \mid FR_T>Q(0.975)]$	1.096	1.072	1.071	1.069	1.175	1.159	1.157
50%-50%-0%	$StDev[FR_T]/E[FR_T]$	0.023	0.007	0.005	0.002	0.058	0.053	0.053
	Q(0.025)	0.964	0.991	0.995	1.000	0.907	0.915	0.916
	Q(0.975)	1.054	1.021	1.016	1.009	1.129	1.119	1.120
	$E[FR_T \mid FR_T<Q(0.025)]$	0.958	0.989	0.993	0.998	0.892	0.903	0.904
	$E[FR_T \mid FR_T>Q(0.975)]$	1.064	1.024	1.019	1.010	1.152	1.138	1.137
37.5%-37.5%-25%	$StDev[FR_T]/E[FR_T]$	0.179	0.177	0.177	0.172	0.176	0.175	0.175
	Q(0.025)	0.825	0.825	0.825	0.832	0.819	0.826	0.826
	Q(0.975)	1.622	1.621	1.619	1.605	1.615	1.602	1.601
	$E[FR_T \mid FR_T<Q(0.025)]$	0.779	0.782	0.782	0.791	0.782	0.787	0.786
	$E[FR_T \mid FR_T>Q(0.975)]$	1.759	1.755	1.755	1.717	1.725	1.717	1.716
25%-25%-50%	$StDev[FR_T]/E[FR_T]$	0.346	0.346	0.345	0.335	0.333	0.331	0.331
	Q(0.025)	0.660	0.660	0.658	0.669	0.667	0.668	0.668
	Q(0.975)	2.398	2.404	2.406	2.381	2.362	2.356	2.356

TABLE 3 – continued

	NL population						
	Micro				Micro + Macro + Parameter		
	500	5000	10000	infinity	500	5000	10000
$E[FR_T \mid FR_T < Q(0.025)]$	0.586	0.587	0.587	0.601	0.604	0.608	0.608
$E[FR_T \mid FR_T > Q(0.975)]$	2.785	2.782	2.782	2.689	2.637	2.626	2.624

The table shows the standard deviation of the funding ratio relative to its expectation, the 2.5% quantile, the 97.5% quantile, and the expected shortfall with respect to these quantiles for an annuity portfolio, which consists of an annuity population portraying the composition of the Dutch population with people older than 24. We report the risk measures for maturity $T = 5$, for several fund sizes (500, 5000, 10,000, and infinitely large fund), and for several (combined) risk sources (micro-, macro-longevity and parameter risk) under alternative investment strategies. The investment strategies are as follows: (a) expected liabilities are cash-flow hedged; (b) liabilities are duration hedged; (c) assets are invested exclusively in 5-year bonds; (d) 50% of the assets is invested into 5-year, and 50% in 10-year bonds; (e) 37.5% is invested into 5-year, 37.5% in 10-year bonds, and the rest is invested into stocks; (f) 25% is invested in 5-year, 25% in 10-year bonds, while the rest is invested in stocks.
Source: Hári et al. (2008b)

The main findings are as follows:

- As the fund size increases, individual mortality risk in relative terms decreases to zero, due to the pooling effect. In contrast, longevity risk does not become negligible; it is almost independent of portfolio size.
- If financial market risk is perfectly hedged (so that uncertainty in future lifetime is the only source of risk), then pension funds are exposed to a substantial amount of uncertainty. For instance, for a large fund (10,000 participants), the standard deviation of the funding ratio in a 5-year horizon is then 5.3% of the expected value.
- If financial market risk is also considered, the contribution of longevity risk to the overall risk becomes less important. However, whenever the investment strategy is not too risky, longevity risk is likely to remain significant.

4.3 *The Ruin Probability*

The approaches discussed in the previous two subsections each have their drawbacks. First, as argued before, quantifying the uncertainty in the discounted present value of liability payments for a given and deterministic interest rate, as in Section 4.1, is a liability only approach that ignores the effect of a pension fund's investment strategy on the impact of longevity risk. In contrast, a funding ratio approach takes into account both assets and liabilities. However, quantifying the uncertainty in the funding ratio, as discussed in Section 4.2, requires making assumptions regarding the fair value of longevity-linked liabilities. As discussed in Section 2.3, it is unlikely that the price of longevity-linked liabilities equals the best estimate value. The best estimate is likely to be an underestimate of the price at which the pension fund could sell its liabilities. One might argue that this problem could be mitigated by adding a *market value margin* to the best estimate value of the liabilities, as suggested by Solvency II. However, when the market value margin does not accurately reflect the risk premium that a third party would require in order to be willing to take over the liabilities, it remains unclear to what extent the funding ratio approach accurately quantifies longevity risk.

In this subsection we discuss an alternative approach to quantify longevity risk, namely by determining its effect on the probability of ruin (see, for example, Olivieri and Pitacco 2003, and Stevens et al. 2010b). We also discuss how this approach relates to, and differs from, the approaches described in the previous two subsections.

Consider again a run-off approach in which there are no new entrants into the fund, and no rights are built up or premiums are paid after time t. Then, for a given (re)investment strategy, the probability of ruin is defined as the probability that the assets available at time t (combined with any future returns on these assets) are insufficient to meet the future liabilities. Specifically, let $t+\mathcal{T}$ denote the last period in which liabilities need to be paid.[22] Then, the probability of ruin is given by $P(A_{t+\mathcal{T}}<0)$, where $A_{t+\mathcal{T}}$ denotes the *terminal asset value,* i.e., the remaining asset value just after the last liability payment has been made (see Olivieri and Pitacco 2003). Longevity risk can be quantified by determining $A_t^{\min}$, the minimum level of the asset value at time t that is required in order to limit the probability of ruin to ϵ. To compare this approach to the approaches described in the two previous subsections, we observe that

$$P\left(A_{t+\mathcal{T}}>0\right)=P\left(A_t>L_t\right), \tag{28}$$

where L_t denotes the date-t present value of future payments, discounted by the portfolio return between date-t and the time of the liability payment, i.e.,

22 In case of pension annuities, we assume that the probability that an individual reaches the age of 111 is negligible, so that $\mathcal{T}=t+110-x_{\min}$, where $x_{\min}$ is the age of the youngest participant in the fund.

$$L_t = \sum_{s=1}^{\mathcal{T}} \frac{\widetilde{L}_{t+s}}{(1+r_t^{(s)})^s}, \tag{29}$$

where $r_t^{(s)}$ denotes the annualized portfolio return over the period $[t, t+s]$. This allows for the following comparison:

- when the asset portfolio consists of one-year bonds and there is no interest rate uncertainty, then $r_t^{(\tau)} = r$, and $L_t = \sum_{s=1}^{\mathcal{T}} \frac{\widetilde{L}_{t+s}}{(1+r)^s}$. Thus, under the assumption that the pension fund will earn a minimal return of r on its investments, the $(1-\epsilon)$-quantile of the discounted present value of liability payments for a constant and deterministic interest rate r, as described in Section 4.1, can be interpreted as the level of assets that is sufficient to guarantee that the probability of ruin is below ϵ.
- Whereas the funding ratio approach described in the previous subsection amounts to comparing, *at a given time* $t+T$, the value of the assets to the *fair value* of the liabilities, the ruin probability approach is equivalent to requiring that the asset value at time t combined with any future returns on these assets is sufficient to cover the *actual* liabilities in each future year.

5 ILLUSTRATING LONGEVITY RISK MANAGEMENT

5.1 *Longevity Risk Management*

As illustrated in the previous section, longevity risk can be substantial for life insurers or pension funds. Likely, this risk factor is not the most important one faced by a life insurer or pension fund, but, given its significance, it cannot be ignored. The typical approach to deal with the effects of changes in mortality rates on pension and insurance liabilities has long been to re-estimate these rates on a regular basis, and to recalculate the value of the liabilities accordingly. Although this accounts to some extent for changes in survival, it is a retrospective approach. It does not take into account future changes in mortality, and thus ignores longevity risk. Instead, a modern risk management approach requires to manage longevity risk, just like other risk factors, in an effective way, see, for instance, Pitacco (2007) or Cairns et al. (2008a). Following Cairns et al. (2008a), there is a range of possibilities to deal with longevity risk.

- Life insurers and pension funds might retain longevity risk as part of their business risk. This would require an appropriate asset liability management (ALM) to guarantee that the assets suffice to meet the liabilities.

As an illustration, we discuss in Section 5.2 the determination of solvency buffers needed to reduce the probability of underfunding of a pension fund or insurance company to an acceptable level.

- Life insurers and pension funds might enter into a variety of forms of reinsurance, or they might arrange a (full or partial) buyout of their liabilities by a specialist insurer. Blake et al. (2008) discuss this traditional possibility in some detail. See also Biffis and Blake (2010).
- Life insurers and pension funds might try to diversify longevity risk, in particular, using different products. Sometimes, natural hedges exist, see, for example, Milevsky and Promislow (2001) or Cox and Lin (2007). We illustrate the diversification possibilities through product mix in Section 5.3. Related to this, in order to share the longevity losses or benefits, life insurers and pension funds might develop new products with adjustable starting dates or payments depending on realized life expectancy.
- Life insurers and pension plans might try to securitize part of their business, or they might try to manage their longevity risk using mortality-linked derivatives. In Section 5.4 we discuss these possibilities further.

5.2 *Solvency Buffers*

In this subsection, we discuss literature regarding the impact of longevity risk on solvency requirements (for example, Olivieri and Pitacco 2003, Hári et al. (2008b), Stevens et al. 2010b). Olivieri and Pitacco 2003 discuss the effect of longevity risk on solvency requirements for life insurers and pension funds. They consider various solvency requirements, each leading to corresponding required asset levels. They illustrate how solvency buffers can be determined, assuming that the one-year death probabilities can be described by the Heligman-Pollard mortality law. Longevity risk arises from three possible future scenarios (a worst case, a medium case, and a best case). Hári et al. (2008b) focus on the probability of underfunding, and determine corresponding solvency buffers in a framework where the uncertainty in the probability distribution of future mortality is quantified by means of the Lee and Carter (1992)-approach. Olivieri and Pitacco (2008) discuss the effect of longevity risk and solvency requirements in relation to Solvency II.

In this subsection we illustrate the determination of solvency buffers in a framework where the uncertainty in the probability distribution of future mortality is quantified by means of the Lee and Carter (1992)-approach. First, we summarize the approach and results in Hári et al. (2008b), who determine solvency buffers on the basis of funding ratio constraints. Next, we summarize the approach in Stevens et al. 2010b), who determine solvency buffers on the basis of ruin probability constraints. In both cases, the buffer is defined as the asset value in excess of the *best estimate* value of the liabilities, and is expressed as a percentage of that best estimate value. Thus, the value

TABLE 4 – CALIBRATED SOLVENCY BUFFER, VaR

T	N	Micro (%)	Micro + Macro (%)	Micro + Macro + Parameter (%)
T=1	500	1.455	2.624	3.760
	1000	1.086	2.412	3.671
	2500	0.723	2.256	3.582
	5000	0.497	2.210	3.515
	10000	0.358	2.179	3.485
T=5	500	3.163	5.178	8.016
	1000	2.331	4.826	7.618
	2500	1.509	4.486	7.282
	5000	1.056	4.269	7.281
	10000	0.774	4.238	7.172

The table presents the percentage c of the best estimate value that has to be invested in a 1- or 5-year bond (depending on the maturity) in order to meet the Value-at-Risk solvency requirement with $\varepsilon = 0.025$, with several (combined) risk sources (micro-, macro-longevity and parameter risk).
Source: Hári et al. (2008b)

of the buffer at time t equals $B_t = c \times BEL_t$, and the goal is to determine the value of c such that:

$$A_t^{\min} = (1 + c) \times BEL_t, \tag{30}$$

where $A_t^{\min}$ again denotes the minimum required level of assets at time t in order to meet the solvency requirement.

Hári et al. (2008b) determine the size of the buffer (i.e., the value of c) required to reduce the probability of underfunding at time $t + T$ to an acceptable level, for a portfolio of deferred annuities, and for a given investment strategy. Specifically, they determine the buffer percentage c such that the Value-at-Risk at the $(1 - \varepsilon) \times 100\%$ level of the funding ratio at time $t + T$ is equal to one, for $T = 1$ and $T = 5$, respectively. To concentrate on mortality risk, all other uncertainties are filtered out. Specifically, they assume that the best estimate value is cash-flow matched at date t, that the buffer is invested in a T-period risk-free zero-coupon bond, and that the term structure of interest rates moves deterministically. They consider pension funds with different sizes, with fund characteristics as defined in Sect 4.2. Table 4 presents the buffer percentage c that is required to meet the solvency requirement with $\varepsilon = 0.025$.

Table 4 illustrates the importance of micro-longevity, macro-longevity, and parameter risk. The combination of micro- and macro-longevity risk implies that a large pension fund has to reserve 4.2% of the best estimate value of the liabilities to meet the solvency requirement in a 5-year horizon. Smaller funds have to reserve even more due to the extra randomness related to micro-longevity risk. If parameter risk is included in the analysis, the initial funding ratio for large funds then has to be 107.2% in order to meet the solvency requirement. It should be noted though that, in view of the results discussed in Section 4.2, it is to be expected that these results could change significantly when investment risk is not filtered out.

Next, we discuss results from Stevens et al. (2010b), who determine the minimal value of the buffer percentage c that is needed to limit the probability of ruin to ε, where the probability of ruin is as defined in Section 4.3. Because pension funds typically also offer partner pensions, they consider two types of liabilities. The first of these is a normalized old-age pension annuity for a 65-year old retiree. The annuity pays off one Euro in every year that the retiree is still alive. Second, they consider a *partner pension* annuity, which consists of a survivor annuity that pays off one Euro in each year that the partner outlives the participant. The authors assume that the portfolio size is sufficiently large for individual mortality risk to be negligible. Moreover, in order to focus on longevity risk, they consider the case in which the best estimate value is invested in a zero-coupon bond portfolio that matches the expected liability payments in each future period. The buffer is (re)invested in one-year zero-coupon bonds. Given this investment strategy, they obtain the following minimal buffer percentages c (as a percentage of the best estimate value) in case $\varepsilon = 2.5\%$, and for 65 year old individuals:

$$
\begin{aligned}
c &= 4.4\%, && \text{for male old-age pension;}\\
&= 12.9\%, && \text{for male partner pension;}\\
&= 4.9\%, && \text{for female old-age pension;}\\
&= 24.6\%, && \text{for female partner pension.}
\end{aligned}
$$

Required buffers are substantial, and significantly higher for partner pension liabilities than for old-age pension liabilities. This occurs because partner pension payments occur later in time, and are therefore more sensitive to longevity risk.[23]

5.3 *The Effect of Portfolio Composition and Product Design*

An obvious way to deal with longevity risk is to try to diversify it using existing products or by developing appropriate new products. In case of new products, the challenge is to identify ways or interventions that limit the

23 These required buffers are substantially lower than those reported by Hári et al. (2008b) for $T = 5$. This occurs due to two reasons: first, whereas Stevens et al. (2010b) consider portfolios of 65-year olds, the population considered in Hári et al. (2008b) contains also younger insured individuals. Annuity payments are more sensitive to longevity risk for younger insured individuals. Second, different models are used to forecast future survival probabilities. The model used in Hári et al. (2008b) induces a higher level of longevity risk.

adverse effects of longevity risk imposed on social security providers, pension funds, and life insurers, and at the same time maintain an adequate level of retirement and life insurance benefits. One of the most heavily debated interventions at the moment, at least in the Netherlands, is an increase in retirement age. In addition, pension funds and insurers can try to redesign characteristics of pension and insurance deals in order to reduce the sensitivity to longevity risk. For example, longevity risk can be affected through strategic choices with regard to the types of insurance that are offered. Whereas old-age pension liabilities are sensitive to long-life risk (the risk that individuals live longer than anticipated), the opposite holds for death benefit insurance, which is adversely affected by short-life risk. This implies that portfolios of death benefit contracts can provide a natural hedge for the mortality risk in old-age pension liabilities (see, for example, Milevsky and Promislow 2001, Cox and Lin 2007). In addition, there may be hedge potential from combining old-age pension annuities and partner pension annuities. Many defined benefit pension funds offer both old-age pension insurance and partner pension insurance.[24] The former consists of a *single life annuity* for the life of the participant. The latter consists of a *survivor annuity* for the life of the partner, if the partner outlives the participant. Stevens et al. (2010b) investigate the effect of product and gender mix on longevity risk, as measured by the probability of ruin, for portfolios consisting of old-age pension liabilities and partner pension liabilities. Their analysis reveals that:

- both product and gender mix can significantly affect longevity risk;
- in general, partner pension liabilities provide natural hedge potential for old-age pension liabilities;
- portfolios with a mixture of male and female insured individuals are typically less sensitive to longevity risk than are portfolios consisting predominantly of males or females.

These results indicate that "unbalanced pension funds" may improve their risk position by engaging in mutual reinsurance.

Another crucial design aspect of pension plans that offer both old-age pension insurance and partner pension insurance is the way in which pension rights are accrued. Two alternatives exist in the Netherlands. In a *JointLife* plan,[25] the participant accrues both old-age pension rights and partner pension rights (i.e., he or she builds up the right to receive the combination of a single life annuity and a survivor annuity). This combination is referred to as a *joint and survivor annuity*. To prevent discrimination between participants with and without a partner, the participant has the option of exchanging,

24 The Retirement Equity Act of 1984 (REA) amended the Employee Retirement Income Security Act of 1974 (ERISA) to introduce mandatory spousal rights in pension plans.

25 In the Netherlands "nabestaandenpensioen op opbouwbasis."

at retirement date, this annuity for a single life annuity that provides higher old-age pension payments. In a *SingleLife* plan,[26] the participant accrues only old-age pension rights (i.e., he or she builds up the right to receive a single life annuity). At retirement date, the participant has the option to exchange this annuity for a joint and survivor annuity that provides both old-age pension insurance and partner pension insurance. In both types of plans, the *conversion rate* (i.e., the rate at which the participant will be able to exchange one type of annuity for the other type), has to be actuarially neutral at the time of exchange. Actuarial neutrality requires that the expected present value of the liabilities before exchange equals the expected present value of the liabilities after exchange. Stevens et al. (2010a) investigate the effect of product design on longevity risk. Their figure (copied here as Figure 7) displays the relative standard deviation of the discounted present value of the liabilities, as defined in (29) for a deterministic interest rate $r = 4\%$, as a function of the age of the participant, for the two pension plans, and for two types of insured individuals: an insured person who will choose a single life annuity at retirement date, and a couple that will choose a joint and survivor annuity in case they are both alive. The old-age pension right is normalized to 1; the partner pension right is normalized to 2/3. The left hand panel is for males and the right hand panel for females.

It can be seen that in both types of plans, longevity risk is substantially lower for a participant who prefers a joint and survivor annuity than for a participant who prefers a single life annuity. Moreover, for both choices, longevity risk is substantially lower in a *JointLife* plan than in a *SingleLife* plan.

5.4 *Securitization and Mortality-Linked Derivatives*

While redesign of pension and insurance deals as illustrated in the previous subsection can mitigate the effects of mortality risk to some extent, this risk will never be eliminated completely. Therefore, uncertainty regarding future survival rates will continue to impose risk on any pension fund or life insurance company.

The introduction of financial instruments for which the payoff is linked, to some extent, to the development of mortality rates could help insurers and pension funds manage their risk. Loeys et al. (2007) investigate whether a "life market," where such products might be traded, could be successful. They explain that for a new capital market to be established and to succeed, "it (1) must provide *effective exposure*, or hedging to a state of the world that is (2) *economically important* and that (3) *cannot be hedged through existing market instruments*, and (4) it must use a *homogeneous and transparent*

26 In the Netherlands "nabestaandenpensioen op risicobasis."

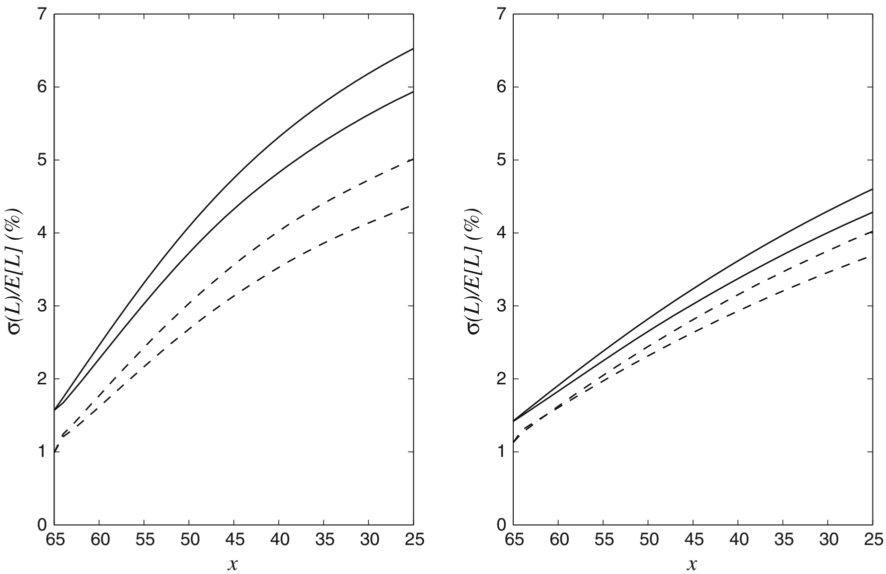

Figure 7 – **Longevity risk and pension plan design.** $\sigma(L)/\mathbb{E}[L]$ as a function of age x for participants without a partner and with a partner who chooses for a singe life annuity at retirement (*solid lines*), and participants with a partner who chooses for a joint and survivor annuity at retirement (*dashed lines*), for a participant in a *JointLife* plan (*lower graphs*) and for a participant in a *SingleLife* plan (*upper graphs*). *Left panel*: males; *right panel*: females. In each case, the old-age pension right is normalized to 1 and the partner pension right is normalized to 2/3. *Source:* Stevens et al. (2010a)

contract to permit exchange between agents." They argue that "longevity meets the basic conditions for a successful market innovation." Blake et al. (2008) investigate the conditions suggested by Loeys et al. (2007) in more detail. These authors maintain that there is insufficient reinsurance capacity to deal with global longevity risk, while capital markets are more efficient than the insurance industry in reducing informational asymmetries and in facilitating price discovery. This makes them confident that a fully developed capital market will emerge soon.

One of the first attempts to set up such a market is a standard coupon-plus-principal bond for which the coupon is determined by the term structure of interest rates, but the principal of the bond depends on the extent to which the actual observed survival in a predefined population, measured by a "survival index," exceeds a given threshold level (see Blake and Burrows 2001, and Blake et al. 2006). By investing in such *longevity bonds*, the risk of higher than expected survival can be partially transferred to the issuer of the bond. The European Investment Bank (EIB) together with BNP Paribas issued a longevity bond in 2004, but there was too little demand to reach a level adequate enough to sustain in a market.[27] The high degree of market incompleteness implies that pricing this product is nontrivial. This might explain why longevity bonds have not yet been successfully introduced in the market. See Blake et al. (2008) for an extensive investigation of this failure.

Indeed, the potentially severe consequences of underpricing the risk may hamper the introduction of longevity linked securities. For example, inaccurate pricing of the risk in guaranteed annuity options induced by uncertain changes in interest rates, led to the downfall of the large British insurance company Equitable in 2000 (see Pelsser 2003). The current literature devotes considerable attention to this pricing problem. Specifically, traditional finance approaches (risk-neutral pricing theories, see, for example, Cairns et al. 2006a) as well as actuarial pricing approaches (Wang's premium principle, see, for example, Lin and Cox 2005) have received considerable attention. However, market incompleteness implies that calibrating these pricing models remains difficult.

An alternative and more successful attempt to deal with longevity risk is securitization.[28] In this case, a pool of assets or liabilities is sold to a so-called Special Purpose Vehicle. These assets or liabilities are then repackaged as new securities, and as such traded in the capital market. Blake et al. (2008) discuss the different types of securitization with longevity-linked assets or liabilities, known as insurance-linked securities (see Krutov 2006). Cowley and Cummins (2005) discuss the earlier types of securitization.

In order to encourage the development of a "life market" JPMorgan introduced in March 2007 so-called "longevity indices." The idea of introducing such indices is that this objective information provided by the indices might stimulate the introduction and subsequent trade of mortality-linked securities.

The mixed success thus far of initiating a life market has generated several proposals to set up such a market using mortality-linked *derivatives*. Mortality and survivor swaps are an example of such derivatives. In case of such a swap one party pays fixed payments to the other party in exchange for payments that depend on the number of people in a given cohort that die in a given period (mortality swap) or that survive during that period (survivor swap), see Dowd et al. (2006) or Dawson et al. (2007). Another example can be found in mortality and longevity forwards. In this case the contract involves the exchange of a payment depending on the realized mortality or survival rate at the maturity of the contract in return for a payment depending on a fixed mortality or survival rate agreed upon at the initiation of the contract. See Blake et al. (2008) for a further discussion and illustration.

27 On the other hand, the issue of short-dated mortality bonds, which are similar to catastrophic bonds, has been successful. However, such bonds hedge against catastrophic mortality risk, not longevity risk.

28 We follow Blake et al. (2008).

Once a market for mortality-linked instruments arises, individual pension funds and insurers can use these instruments to hedge or reduce their risk. The asset portfolio should be designed in order to yield an optimal risk-return trade-off. See, for instance, Haberman and Vigna (2002) or Gerrard et al. (2004). A concern here is that optimization of risk-return trade-offs is known to be highly sensitive to parameter estimation if parameter uncertainty is ignored (for example, Best and Grauer 1991, or Chopra and Ziemba 1993). This is clearly undesirable if the optimal portfolios with respect to the estimated parameter values lead to significantly suboptimal values of the objective function compared to the optimal portfolios for the true parameter values. In particular, this is a major concern in a setting in which mortality-linked assets and liabilities are involved, since the probability distribution of their value depends on very long-term forecasts. This requires the development of robust portfolio optimization techniques for Asset Liability Management of an individual pension fund or insurer in the presence of mortality risk.

6 CONCLUSIONS

This paper investigates longevity risk, i.e., the uncertainty in future changes in mortality rates. We illustrate the importance of longevity risk for pension funds and life insurers, and we illustrate some aspects of longevity risk management, in particular, the determination of solvency buffers, and the effect of the product mix as a natural approach to diversifying the longevity risk. We also briefly discuss the attempts to set up a "life market," a trading place for mortality-based products, that could be used to hedge or to reduce the longevity risk. These initiatives have thus far been only partially successful, even when "longevity meets the basic conditions for a successful market innovation" (Loeys et al. 2007). As discussed by Blake et al. (2008) the government might assist by encouraging and facilitating the development of this market. In particular, the government could issue longevity bonds in order to establish a default-free term structure for longevity risk, similar to its activity in the fixed-income market.

Let us conclude by indicating some directions for future research. This paper focusses on the effect of longevity risk on (pension) annuities. However, increased longevity will likely also have non-negligible effects on health care costs. For example, greater numbers of older people in society will obviously increase the burden on health care systems, because older people on average need more health care. Second, the health policy literature extensively documents trends in health status as a function of age (see, for example, Murray and Lopez 1997, and Stallard 2005). These trends may affect the per capita costs for health care, which is particularly important to health and disability insurers. The mortality models discussed in Section 3 provide a purely probabilistic description of the future development of the survival probabilities. In particular, these models do not account for the interaction between mortality and morbidity. In contrast, the demographic and health policy literature did develop models that account for the interaction between morbidity and mortality. Examples are Murray and Lopez (1997), Kytir and Prskawetz (1995), and Manton et al. (1994). These approaches, however, provide deterministic projections rather than stochastic forecasts. We would argue that there is a need to develop models to jointly forecast mortality and morbidity rates, as a function of age and gender, accounting properly for dependence between mortality and morbidity, and for the degree of uncertainty inherent in the forecasts.

Finally, we have assumed that the development in future mortality is independent of the development in other economic quantities. However, as is illustrated, for instance, by the Preston curve (see, for example, Preston 2007), there is a clear correlation between the average life expectancy in a given country and this country's welfare level (measured by GDP per capita). This suggests additional ways to hedge longevity risk, a relevant topic deserving further research.

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